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THE SOLUTION OF THE ILL-POSED CAUCHY PROBLEM FOR MATRIX FACTORIZATIONS OF THE HELMHOLTZ EQUATION

Juraev Davron Aslonqulovich*

Department of Natural Science Disciplines, Higher Military Aviation School of the Republic of Uzbekistan, Karshi, Uzbekistan

Abstract. In the paper it is considered the problem of regularization of the Cauchy problem for matrix factorisations of the Helmholtz equation in three-dimensional unbounded domain. Using the Carleman matrix found an explicitly regularized solution of the Cauchy problem for matrix factorizations of the Helmholtz equation in three-dimensional unbounded domain.

Keywords: Cauchy problem, regularization, factorization, regular solution, fundamental solution.

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Corresponding author: Juraev Davron Aslonqulovich, Associate Professor at the Department of Natural Science Disciplines, Higher Military Aviation School of the Republic of Uzbekistan, Jaykhun St., Kashkadarya Region, Karshi, 180200, Uzbekistan, Tel.: +99891 9554712, e-mail: *juraev_davron@list.ru*

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1 Introduction

This problem concerns ill-posed problems, i.e., it is unstable. It is known that the Cauchy problem for elliptic equations is unstable relatively small change in the data, i.e., incorrect (example Hadamard, see for instance Adamar, 1978). There is a sizable literature on the subject (Adamar, 1978; Aizenberg, 1990; Aizenberg at al., 1988; Goluzin at al., 1933; Carleman, 1926; Lavrent'ev, 1957; Lavrent'ev, 1962; Tikhonov, 1963; Yarmukhamedov, 1977; Yarmukhamedov, 1997; Yarmukhamedov, 2004). N.N. Tarkhanov (1989) has published a criterion for the solvability of a larger class of boundary value problems for elliptic systems. In unstable problems, the image of the operator is not is closed, therefore, the solvability condition can not be is written in terms of continuous linear functionals. So, in the Cauchy problem for elliptic equations with data on part of the boundary of the domain the solution is usually unique, the problem is solvable for everywhere dense a set of data, but this set is not closed. Consequently, the theory of solvability of such problems is much more difficult and deeper than theory of solvability of Fredholm equations. The first results in this direction appeared only in the mid-1980s in the works of L.A. Aizenberg, A.M. Kytmanov, N.N. Tarkhanov (see, for instance Tarkhanov, 1995).

The uniqueness of the solution follows from Holmgren's general theorem (see Bers at al., 1966). The conditional stability of the problem follows from the work of A.N. Tikhonov (Tikhonov, 1963), if we restrict the class of possible solutions to a compactum.

In this paper we construct a family of vector-functions $U_{\sigma(\delta)}(x) = U(x, f_\delta)$ depending on a parameter σ, and prove that under certain conditions and a special choice of the parameter $\sigma = \sigma(\delta)$, at $\delta \to 0$, the family $U_{\sigma(\delta)}(x)$ converges in the usual sense to a solution $U(x)$ at a point $x \in G$.

Following A.N. Tikhonov (Tikhonov, 1963), a family of vector-valued functions $U_{\sigma(\delta)}(x)$ is called a regularized solution of the problem. A regularized solution determines a stable method of approximate solution of the problem. For special domains, the problem of extending bounded analytic functions in the case when the data are given only on a part of the boundary was considered by Carleman (Carleman, 1926). The researches of T. Carleman were continued by G.M. Goluzin and V.I. Krylov (Goluzin at al., 1933). A multidimensional analogue of Carleman's formula for analytic functions of several variables was constructed in (Aizenberg, 1990). The use of the classical Green's formula for constructing a regularized solution of the Cauchy problem for the Laplace equation was proposed by Academician M.M. Lavrent'ev (Lavrent'ev, 1957; Lavrent'ev, 1962). Using the ideas of M.M. Lavrent'ev and Sh. Yarmukhamedov, a regularized solution of the Cauchy problem for the Laplace and Helmholtz equations was constructed in explicit form (Yarmukhamedov, 1977; Yarmukhamedov, 1997; Yarmukhamedov, 2004). In work (Tarkhanov, 1985) an integral formula is proved for systems of equations of elliptic type of the first order, with constant coefficients in a bounded domain. In work (Arbuzov at al., 2006), the Cauchy problem for the Helmholtz equation in an arbitrary bounded plane domain with Cauchy data, known only on the region boundary, is considered. The solvability criterion for the Cauchy problem for the Laplace equation in the space $\mathbb{R}^m$ it was considered by Shlapunov in work (Shlapunov, 1992). In work (Kabanikhin at al., 2013), was be continuation the problem for the Helmholtz equation is investigated and the results of numerical experiments are presented.

The construction of the Carleman matrix for elliptic systems was carried out by: Sh. Yarmukhamedov, N.N. Tarkhanov, A.A. Shlapunov, I.E. Niyozov, D.A. Juraev and others. The system considered in this paper was introduced by N.N. Tarkhanov. For this system, he studied correct boundary value problems and found an analogue of the Cauchy integral formula in a bounded domain.

The system considered in this paper was introduced by N.N. Tarkhanov. For this system, he studied correct boundary value problems and found an analogue of the Cauchy integral formula in a bounded domain (Tarkhanov, 1995).

For systems of equations of elliptic type of the first order with constant coefficients, the factorizing operator of Helmholtz, in work (Juraev, 2017b) the validity of the integral formula in a three-dimensional unbounded domain was proved.

In many well-posed problems for systems of equations of elliptic type of the first order with constant coefficients that factorize the Helmholtz operator, it is not possible to calculate the values of the vector function on the entire boundary. Therefore, the problem of reconstructing the solution of systems of equations of first order elliptic type with constant coefficients, factorizing the Helmholtz operator (Juraev, 2012; Juraev, 2014; Juraev, 2017a; Juraev, 2017b; Juraev, 2017c; Juraev, 2018a; Juraev, 2018b; Juraev, 2018c; Juraev, 2019a; Juraev, 2019b), is one of the topical problems in the theory of differential equations.

For the last decades, interest in classical ill-posed problems of mathematical physics has remained. This direction in the study of the properties of solutions of the Cauchy problem for the Laplace equation was started in works (Lavrent'ev, 1957; Lavrent'ev, 1962; Yarmukhamedov, 1977; Adamar, 1978), and subsequently developed in works (Tarkhanov, 1985; Tarkhanov, 1989; Tarkhanov, 1995; Shlapunov, 1992; Shlapunov, 2011; Niyozov, 2014; Juraev, 2012; Juraev, 2014; Juraev, 2017a; Juraev, 2017b; Juraev, 2017c; Juraev, 2018a; Juraev, 2018b; Juraev, 2018c; Juraev, 2019a; Juraev, 2019b).

In this paper, we present an explicit formula for the approximate solution of the Cauchy problem for the matrix factorizations of the Helmholtz equation in an unbounded region on the space. Our formula for an approximate solution also includes the construction of a family of fundamental solutions for the Helmholtz operator on the space. This family is parametrized by some entire function $K(w)$, the choice of which depends on the dimension of the space. In this work, based on the results of (Juraev, 2017b), similarly we obtain better results by choosing the function $K(w)$.

Let $\mathbb{R}^3$ be the three-dimensional real Euclidean space,

$$x = (x_1, x_2, x_3) \in \mathbb{R}^3,\ y = (y_1, y_2, y_3) \in \mathbb{R}^3,$$

$$x' = (x_1, x_2) \in \mathbb{R}^2,\ y' = (y_1, y_2) \in \mathbb{R}^2.$$

$G \subset \mathbb{R}^3$ be an unbounded simply connected domain with a piecewise-smooth boundary consisting of the plane T: $y_3 = 0$ and a smooth surface S lying in the half-space $y_3 > 0$, i.e., $\partial G = S \bigcup T$.

We introduce the following notation:

$$r = |y - x|\,,\ \alpha = \left|y' - x'\right|, w = i\sqrt{u^2 + \alpha^2} + y_3,\ u \geq 0,$$

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)^T, \frac{\partial}{\partial x} \to \xi^T,\ \xi^T = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} - \text{transposed vector } \xi,$$

$$U(x) = (U_1(x), \ldots, U_n(x))^T,\ u^0 = (1, \ldots, 1) \in \mathbb{R}^n,\ n = 2^m,\ m = 3,$$

$$E(z) = \left\| \begin{matrix} z_1 \ldots 0 \\ \ldots\ldots. \\ 0 \ldots z_n \end{matrix} \right\| - \text{the diagonal matrix, } z = (z_1, \ldots, z_n) \in \mathbb{R}^n.$$

Let $D(\xi^T)$, $(n \times n)-$ dimensional matrix with elements consisting of a set of linear functions with constant coefficients of the complex plane for which the following condition is satisfied:

$$D^*(\xi^T)D(\xi^T) = E((|\xi|^2 + \lambda^2)u^0),$$

where $D^*(\xi^T)$ is the Hermitian conjugate matrix of $D(\xi^T)$, λ is a real number.

Consider in the domain G the system of differential equations

$$D\left(\frac{\partial}{\partial x}\right)U(x) = 0, \tag{1}$$

where $D\left(\frac{\partial}{\partial x}\right)$ is the matrix of first-order differential operators.

We denote by $A(G)-$ the class of vector functions in the domain G continuous on $\overline{G} = G \bigcup \partial G$ and satisfying system (1).

If G is a bounded domain and $U(y) \in A(G)$, then the following integral formula of Cauchy type is true (Juraev, 2014; Juraev, 2017a; Juraev, 2017b; Juraev, 2017c; Juraev, 2018a; Juraev, 2018b; Juraev, 2018c; Juraev, 2019a; Juraev, 2019b)

$$U(x) = \int\limits_{\partial G} N(y, x)U(y)ds_y,\ \ x \in G, \tag{2}$$

where

$$N(y, x) = \left(E\left(-\frac{e^{i\lambda r}}{4\pi r}u^0\right)D^*\left(\frac{\partial}{\partial y}\right)\right)D(t^T).$$

Here $t = (t_1, t_2, t_3)$ is the unit external normal drawn at the point y, the surface ∂G, $-\dfrac{e^{i\lambda r}}{4\pi r}$ is a fundamental solution to the Helmholtz equation in $\mathbb{R}^3$. (Bers at al.,1966; Aleksidze, 1991; Juraev, 2012).

We denote by $K(w)-$ the entire function taking real values for real w ($w = v + iu$, u, $v-$ is real numbers) and satisfying the conditions:

$$K(u) \neq 0,\ \sup_{v \geq 1}\left|\xi^p K^{(p)}(w)\right| = M(u, p) < \infty,\ \ -\infty < u < \infty,\ p = 0, 1, 2, 3.$$

The function $\Phi(y, x)$ at $y \neq x$ is defined by the following equality:

$$\Phi(y,x) = -\frac{1}{2\pi^2 K(x_3)} \int_0^\infty \mathrm{Im} \frac{K(w)}{w - x_3} \frac{\cos \lambda u}{\sqrt{u^2 + \alpha^2}} du. \tag{3}$$

Formula (2) is true if instead of $-\dfrac{e^{i\lambda r}}{4\pi r}$ we substitute the function

$$\Phi(y,x) = -\frac{e^{i\lambda r}}{4\pi r} + g(y,x), \tag{4}$$

where $g(y, x)$ is the regular solution of the Helmholtz equation with respect to the variable y, including the point $y = x$.

Then formula (2) has the following form

$$U(x) = \int_{\partial G} N(y,x) U(y) ds_y, \quad x \in G, \tag{5}$$

where

$$N(y,x) = \left(E\left(\Phi(y,x)u^0\right) D^*\left(\frac{\partial}{\partial y}\right)\right) D(t^T).$$

2 The integral formula in an unbounded domain

We generalize formula (5) for the case when $G-$ is an unbounded domain (Juraev, 2017b).

Let $G \subset \mathbb{R}^3$ be an unbounded domain with a piecewise-smooth boundary ∂G ($\partial G-$ extends to infinity).

We denote by G_R the part of G lying inside the circle of radius R centered at zero:

$$G_R = \{y : y \in G, \ |y| < R\}, G_R^\infty = G \backslash G_R, R > 0.$$

Theorem 1. *Let $U(y) \in A(G)$, $G-$be a simply connected unbounded domain in $\mathbb{R}^3$ with a piecewise-smooth boundary ∂G. If for each fixed $x \in G$, holds the equality*

$$\lim_{R\to\infty} \int_{G_R^\infty} M(y,x) U(y) ds_y = 0, \tag{6}$$

then the following formula is true

$$U(x) = \int_{\partial G} N(y,x) U(y) ds_y, \quad x \in G, \tag{7}$$

where

$$N(y,x) = \left(E\left(\Phi(y,x)u^0\right) D^*\left(\frac{\partial}{\partial y}\right)\right) D(t^T).$$

Proof. Indeed, for a fixed $x \in G$ ($|x| < R$) and taking into account (6), we have

$$\int_{\partial G} N(y,x) U(y) ds_y = \int_{\partial G_R} N(y,x) U(y) ds_y +$$

$$+ \int_{\partial G_R^\infty} N(y,x) U(y) ds_y = U(x) + \int_{\partial G_R^\infty} N(y,x) U(y) ds_y, \ x \in G_R.$$

Now at $R \to \infty$, we obtain the integral formula (7). Theorem 1 is proved. $\square$

We consider particular cases of formula (7). Suppose that an unbounded domain G lies inside the layer of smallest width defined by the inequality

$$0 < y_3 < h,\ h = \frac{\pi}{\rho},\ \rho > 0,$$

moreover ∂G extends to infinity.

Suppose that for some $b_0 > 0$ the area ∂G satisfies the growth condition

$$\int\limits_{\partial G} \exp\left[-b_0\rho_0\left|y'\right|\right] ds_y < \infty,\ 0 < \rho_0 < \rho. \tag{8}$$

Let $U(y) \in A(G)$ satisfy the boundary growth condition

$$|U(y)| \leq \exp\left[\exp \rho_2\left|y'\right|\right],\ \rho_2 < \rho,\ y \in G. \tag{9}$$

In equality (3) we put

$$\begin{gathered} K(w) = \exp\left[-bi\rho_1\left(w - \frac{h}{2}\right) - b_1 i\rho_0\left(w - \frac{h}{2}\right)\right], \\ K(x_3) = \exp\left[b\cos\rho_1\left(x_3 - \frac{h}{2}\right) + b_1\cos i\rho_0\left(x_3 - \frac{h}{2}\right)\right], \\ 0 < \rho_1 < \rho,\ 0 < x_3 < h, \end{gathered} \tag{10}$$

where

$$b = 2a\exp\left(\rho_1\left|x'\right|\right),\ b_1 > \frac{b_0}{\cos\left(\rho_0\dfrac{h}{2}\right)},\ a \geq 0,\ b > 0.$$

Then the integral representation (7) is true.

For a fixed $x \in G$ and $y \to \infty$, we estimate the function $\Phi(y, x)$ and its derivatives $\dfrac{\partial\Phi(y, x)}{\partial y_j}$, $j = 1, 2$ and $\dfrac{\partial\Phi}{\partial y_3}(y, x)$. To estimate $\dfrac{\partial\Phi(y, x)}{\partial y_j}$, we use the equality

$$\frac{\partial\Phi(y, x)}{\partial y_j} = \frac{\partial\Phi(y, x)}{\partial s}\frac{\partial s}{\partial y_j} = 2(y_j - x_j)\frac{\partial\Phi(y, x)}{\partial s},\ j = 1, 2.$$

Really,

$$\begin{gathered} \left|\exp\left[-bi\rho_1\left(w - \frac{h}{2}\right) - b_1 i\rho_0\left(w - \frac{h}{2}\right)\right]\right| = \\ = \exp\operatorname{Re}\left[-bi\rho_1\left(w - \frac{h}{2}\right) - b_1 i\rho_0\left(w - \frac{h}{2}\right)\right] = \\ = \exp\left[-b\rho_1\sqrt{u^2 + \alpha^2}\cos\rho_1\left(y_3 - \frac{h}{2}\right) - b_1\rho_0\sqrt{u^2 + \alpha^2}\cos\rho_0\left(y_3 - \frac{h}{2}\right)\right]. \end{gathered}$$

As

$$\begin{gathered} -\frac{\pi}{2} \leq -\frac{\rho_1}{\rho}\cdot\frac{\pi}{2} \leq \frac{\rho_1}{\rho}\cdot\frac{\pi}{2} < \frac{\pi}{2}, \\ -\frac{\pi}{2} \leq -\frac{\rho_1}{\rho}\cdot\frac{\pi}{2} \leq \rho_0\left(y_3 - \frac{h}{2}\right) \leq \frac{\rho_1}{\rho}\cdot\frac{\pi}{2} < \frac{\pi}{2}. \end{gathered}$$

Hence,

$$\cos\rho\left(y_3 - \frac{h}{2}\right) > 0, \cos\rho_0\left(y_3 - \frac{h}{2}\right) \geq \cos\frac{h\rho_0}{2} > \delta_0 > 0,$$

does not vanish in the domain G and the following asymptotic estimates are true

$$|\Phi(y,x)| = \mathrm{O}\left[\exp\left(-\varepsilon\rho_1\left|y'\right|\right)\right],\ \varepsilon>0,\ y\to\infty,\ y\in G\bigcup\partial G,$$

$$\left|\frac{\partial\Phi(y,x)}{\partial y_j}\right| = \mathrm{O}\left[\exp\left(-\varepsilon\rho_1\left|y'\right|\right)\right],\ \varepsilon>0,\ y\to\infty,\ y\in G\bigcup\partial G,\ j=1,2. \tag{11}$$

$$\left|\frac{\partial\Phi(y,x)}{\partial y_3}\right| = \mathrm{O}\left[\exp\left(-\varepsilon\rho_1\left|y'\right|\right)\right],\ \varepsilon>0,\ y\to\infty,\ y\in G\bigcup\partial G.$$

Now we choose ρ_1 with condition $\rho_2<\rho_1<\rho$. Using the boundary condition (8) and asymptotic estimates (11), we obtain the fulfillment of condition (6). Therefore, the integral formula (7) holds in the an unbounded domain G.

Condition (9) can be relaxed.

We denote by $A_\rho(G)-$ the class of functions from $A(G)$ that satisfy the following growth condition:

$$A_\rho(G) = \left\{U(y):\ U(y)\in A(G),\ \ |U(y)|\le\exp\left[\mathrm{o}\left(\exp\rho\left|y'\right|\right)\right],\ y\to\infty,\ y\in G\right\}. \tag{12}$$

The following is true.

Theorem 2. *Let $U(y)\in A_\rho(G)$ satisfy the growth condition*

$$|U(y)|\le C\exp\left[a\cos\rho_1\left(y_3-\frac{h}{2}\right)\exp\left(\rho_1\left|y'\right|\right)\right],$$
$$a\ge 0,\ 0<\rho_1<\rho,\ y\in\partial G, \tag{13}$$

where $C-$ is some constant. Then formula (6) holds.

Proof. We divide the area G with a line $y_3=\dfrac{h}{2}$ into two areas

$$G_1=\left\{y:\ 0<y_3<\frac{h}{2}\right\}\text{ and } G_2=\left\{y:\ \frac{h}{2}<y_3<h\right\}.$$

Consider the domain G_1. In the formula (4), instead of $K(w)$, we put $K_1(w)$

$$K_1(w)=K(w)\exp\left[-\delta i\tau\left(w-\frac{h}{2}\right)-\delta_1 i\rho\left(w-\frac{h}{2}\right)\right],$$
$$\rho<\tau<2\rho,\ \delta>0,\ \delta_1>o, \tag{14}$$

Here $K(w)$ it is determined from (10). Under these notations, estimate (8) is true.

Really,

$$\left|\exp\left[-i\tau\left(w-\frac{h}{4}\right)-\delta_1 i\rho\left(w-\frac{h}{4}\right)\right]\right|=$$

$$=\exp\left[-\delta\tau\sqrt{u^2+\alpha^2}\cos\tau\left(y_3-\frac{h}{4}\right)\right]=$$

$$=\exp\left[-\delta\tau\sqrt{u^2+\alpha^2}\right]\le\exp\left[-\delta\exp\tau\left|y_1\right|\right].$$

As

$$-\frac{\pi}{2}\le-\tau\frac{\pi}{4}\le\tau\left(y_3-\frac{h}{4}\right)\le\tau\frac{\pi}{2}<\frac{h}{2}\text{ and }\cos\tau\left(y_3-\frac{h}{4}\right)\ge\cos\tau\frac{h}{4}\ge\delta_0>0.$$

We denote the corresponding $\Phi(y,x)$ by $\Phi^+(y,x)$.

Hence,

$$\cos\tau\left(y_3-\frac{h}{4}\right)\geq\delta_0,\ y\in G_1\bigcup\partial G_1,$$

then for fixed $x\in G_1,\ y\in G_1\bigcup\partial G_1$, asymptotic estimates are true for $\Phi^+(y,x)$ and its derivatives

$$\left|\Phi^+(y,x)\right|=\mathrm{O}\left[\exp(-\delta_0\exp\left(\tau\left|y'\right|\right)\right],\ y\to\infty,\ \rho<\tau<2\rho,$$

$$\left|\frac{\partial\Phi^+(y,x)}{\partial y_j}\right|=\mathrm{O}\left[\exp(-\delta_0\exp\left(\tau\left|y'\right|\right)\right],\ y\to\infty,\ \rho<\tau<2\rho,\ j=1,2.$$

$$\left|\frac{\partial\Phi^+(y,x)}{\partial y_3}\right|=\mathrm{O}\left[\exp(-\delta_0\exp\left(\tau\left|y'\right|\right)\right],\ y\to\infty,\ \rho<\tau<2\rho.$$

Let $U(y)\in A(G_1)$ in the domain G_1 satisfy the growth condition

$$|U(y)|\leq C\exp\left[\exp\left(2\rho-\varepsilon\right)\left|y'\right|\right],\ \varepsilon>0. \tag{15}$$

We choose τ in (14) from the inequality $2\rho-\varepsilon<\tau<2\rho$.

Then condition (15) is fulfilled for the domain G_1; therefore, holds the following integral formula

$$U(x)=\int\limits_{\partial G_1}N(y,x)U(y)ds_y,\ x\in G_1, \tag{16}$$

where

$$N(y,x)=\left(E\left(\Phi^+(y,x)u^0\right)D^*\left(\frac{\partial}{\partial y}\right)\right)D(t^T).$$

If $U(y)\in A(G_2)$ satisfies the growth condition (15) in G_2, then for $2\rho-\varepsilon<\tau<2\rho$ similarly we obtain the following integral formula

$$U(x)=\int\limits_{\partial G_2}N(y,x)U(y)ds_y,\ x\in G_2, \tag{17}$$

where

$$N(y,x)=\left(E\left(\Phi^-(y,x)u^0\right)D^*\left(\frac{\partial}{\partial y}\right)\right)D(t^T).$$

Here $\Phi^-(y,x)$ it is determined by formula (3), in which $K(w)$ is replaced by the function $K_2(w)$:

$$K_2(w)=K(w)\exp\left[-\delta i\tau\left(w-h_1\right)-\delta_1 i\rho\left(w-\frac{h}{2}\right)\right], \tag{18}$$

where

$$h_1=\frac{h}{2}+\frac{h}{4},\ \frac{h}{2}<y_3<h,\ \frac{h}{2}<x_3<h_1,\ \delta>0,\ \delta_1>0.$$

The integrals obtained in this formula (according to (13)) uniformly converge at $\delta\geq0$, when $U(y)\in A_\rho(G)$. Put $\delta=0$ in these formulas and, combining the formulas obtained, we find

$$U(x)=\int\limits_{\partial G}N(y,x)U(y)ds_y,\quad x\in G,\ x_3\neq\frac{h}{2}, \tag{19}$$

where

$$N(y,x)=\left(E\left(\tilde{\Phi}(y,x)u^0\right)D^*\left(\frac{\partial}{\partial y}\right)\right)D(t^T).$$

(the integrals over the cross section $y_3=\dfrac{h}{2}$ are mutually annihilated).

$$\tilde{\Phi}(y,x) = (\Phi^+(y,x))_{\delta=0} = (\Phi^-(y,x))_{\delta=0}.$$

Here $\tilde{\Phi}_\sigma(y,x)$ is determined by formula (3), in which $K(w)$ is determined from (14), where $\delta = 0$ is assigned. According to the continuation principle, formula (19) is true for $\forall x \in G$. Under condition (15), formula (19) is true for $\forall \delta_1 \geq 0$. Assuming $\delta_1 = 0$ we get the proof theorems. Theorem 2 is proved. □

3 The Cauchy problem and the construction of the Carleman function

In the formula (3), choosing

$$K(w) = \frac{1}{(w - x_3 + 2h)^2}\exp(\sigma w^2),$$
$$K(x_3) = \frac{1}{(2h)^2}\exp(\sigma x_3^2),\ 0 < x_3 < h,\ h = \frac{\pi}{\rho}, \tag{20}$$

we get

$$\Phi_\sigma(y,x) = -\frac{2h^2 e^{-\sigma x_3^2}}{\pi^2}\int_0^\infty \mathrm{Im}\frac{\exp(\sigma w^2)}{(w - x_3 + 2h)^2(w - x_3)}\frac{\cos\lambda u}{\sqrt{u^2+\alpha^2}}du. \tag{21}$$

Then the integral formula (7) has the following form:

$$U(x) = \int_{\partial G} N_\sigma(y,x)U(y)ds_y, \quad x \in G, \tag{22}$$

where

$$N_\sigma(y,x) = \left(E\left(\Phi_\sigma(y,x)u^0\right)D^*\left(\frac{\partial}{\partial y}\right)\right)D(t^T).$$

Suppose that the boundary of the domain G consists of a hyper plane $y_3 = 0$ and a smooth surface S extending to infinity and lying in a layer

$$0 < y_3 < h,\ h = \frac{\pi}{\rho},\ \rho > 0.$$

We assume that S is given by the equation

$$y_3 = \psi(y_1, y_2),\ -\infty < y_1 < \infty,\ -\infty < y_2 < \infty,$$

where $\psi(y')$ satisfies the condition

$$\left|\frac{\partial\psi(y')}{\partial y_j}\right| \leq M < \infty,\ y' \in \mathbb{R}^2,\ j = 1, 2., M = const.$$

Statement of the problem. Let $U(y) \in A_\rho(G)$ and

$$U(y)|_S = f(y),\ y \in S. \tag{23}$$

Here, $f(y)-$ is a given continuous vector function on S.

It is required to restore the vector function $U(y)$ in the domain G, based on its values of $f(y)$ on S.

The following is true.

Theorem 3. *Let $U(y) \in A_\rho(G)$ satisfy the inequality*

$$|U(y)| \leq M, \ y \in T. \tag{24}$$

If

$$U_\sigma(x) = \int\limits_S N_\sigma(y,x) U(y) ds_y, \quad x \in G. \tag{25}$$

Then the following estimate holds

$$|U(x) - U_\sigma(x)| \leq C_\rho(x) M \sigma e^{-\sigma x_3^2}, \ \sigma > 1, \ x \in G. \tag{26}$$

Here and below functions bounded on compact subsets of the domain G, we denote by $C_\rho(x)$. Moreover, they are different in various inequalities.

Proof. Using the integral formula (22) and equality (25), we obtain

$$U(x) = U_\sigma(x) + \int\limits_T N_\sigma(y,x) U(y) ds_y, \ x \in G.$$

Given inequality (24), we estimate the following

$$|U(x) - U_\sigma(x)| \leq \int\limits_T |U(y)| \left|N_\sigma(y,x)\right| ds_y \leq M \int\limits_T \left|N_\sigma(y,x)\right| ds_y, \ x \in G.$$

To do this, we estimate the integrals $\int\limits_T |\Phi_\sigma(y,x)| \, ds_y$, $\int\limits_T \left|\frac{\partial \Phi_\sigma(y,x)}{\partial y_j}\right| ds_y$, $j = 1,2$ and $\int\limits_T \left|\frac{\partial \Phi_\sigma}{\partial y_3}(y,x)\right| ds_y$ on the part T of the plane $y_3 = 0$.

Let $\sigma > 0$. Separating the imaginary part of equality (21), we obtain

$$\Phi_\sigma(y,x) = \frac{2h^2 e^{\sigma(y_3^2 - x_3^2)}}{\pi^2} \left[\int\limits_0^\infty \left(\frac{e^{-\sigma(u^2+\alpha^2)} \left(-\alpha_1^2 + \beta_1^2 + 2\beta_1\beta\right) \cos\gamma\alpha_1}{\left(\alpha_1^2 + \beta_1^2\right)^2 \left(\alpha_1^2 + \beta^2\right)} + \right.\right.$$
$$\left.\left. + \frac{e^{-\sigma(u^2+\alpha^2)} \left(2\alpha_1^2\beta_1 + \alpha_1^2\beta - \beta_1^2\beta\right)}{\left(\alpha_1^2 + \beta_1^2\right)^2 \left(\alpha_1^2 + \beta^2\right)} \frac{\sin\gamma\alpha_1}{\alpha_1} \right) \cos\lambda u \, du \right], \tag{27}$$

where

$$\gamma = 2\sigma y_3, \ \alpha_1^2 = u^2 + \alpha^2, \ \beta = y_3 - x_3, \ \beta_1 = y_3 - x_3 + 2h.$$

We use easily verified inequalities

$$\left|\frac{d^k}{ds^k} \frac{\sin\gamma\alpha_1}{\alpha_1}\right| \leq C_k \frac{\gamma^k}{\alpha_1^{k+1}}, \ \alpha_1 \geq 1, \ k = 0, 1, \ldots,$$

$$\left|\frac{d^k}{ds^k} \frac{\sin\gamma\alpha_1}{\alpha_1}\right| \leq C_k \frac{\gamma^k}{\alpha_1^{2k}}, \ 0 < \alpha_1 \leq 1,$$

$$\left|\frac{d^k}{ds^k} \cos\gamma\alpha_1\right| \leq C_k \frac{\gamma^k}{\alpha_1^k}, \ \alpha_1 \geq 1, \ k = 0, 1, \ldots, \tag{28}$$

$$\left|\frac{d^k}{ds^k} \cos\gamma\alpha_1\right| \leq C_k \frac{\gamma^k}{\alpha_1^{2(k-1)}}, \ 0 < \alpha_1 \leq 1, \ k = 2, \ldots,$$

$$\left|\frac{d}{ds} \cos\gamma\alpha_1\right| \leq \frac{\gamma}{\alpha_1}, \ \gamma > 0, \ \gamma = 2\sigma y_3, \ C = const.$$

We first estimate $\int\limits_{y_3=0} |\Phi_\sigma(y,x)|\,ds_y$. Given equality (27) and inequality (28), for $\alpha_1 \geq 1$ and $0 < \alpha_1 \leq 1$, we have

$$\int\limits_{y_3=0} |\Phi_\sigma(y,x)|\,ds_y \leq C_\rho(x)\sigma\, e^{-\sigma x_3^2},\ \sigma > 1,\ x \in G, \tag{29}$$

$$\int\limits_{y_3=0} |\Phi_\sigma(y,x)|\,ds_y \leq C_\rho(x)\sigma\, e^{-\sigma x_3^2},\ \sigma > 1,\ x \in G. \tag{30}$$

Combining inequalities (29) - (30), we obtain

$$\int\limits_{y_3=0} |\Phi_\sigma(y,x)|\,ds_y \leq C_\rho(x)\sigma\, e^{-\sigma x_3^2},\ \sigma > 1,\ x \in G. \tag{31}$$

To estimate the second integral, we use the equality

$$\frac{\partial \Phi_\sigma(y,x)}{\partial y_j} = \frac{\partial \Phi_\sigma(y,x)}{\partial s}\frac{\partial s}{\partial y_j} = 2(y_j - x_j)\frac{\partial \Phi_\sigma(y,x)}{\partial s},\ j = 1, 2. \tag{32}$$

Where

$$\frac{\partial \Phi_\sigma(y,x)}{\partial s} = \frac{2h^2 e^{\sigma(y_3^2 - x_3^2)}}{\pi^2}\left[\int\limits_0^\infty e^{-\sigma(u^2+\alpha^2)}\left(\frac{-\sigma(-\alpha_1^2+\beta_1^2+2\beta_1\beta)\cos\gamma\alpha_1}{(\alpha_1^2+\beta_1^2)^2(\alpha_1^2+\beta^2)} - \right.\right.$$

$$-\frac{\cos\gamma\alpha_1}{(\alpha_1^2+\beta_1^2)^2(\alpha_1^2+\beta^2)} - \frac{\sigma y_3(-\alpha_1^2+\beta_1^2+2\beta_1\beta)}{(\alpha_1^2+\beta_1^2)^2(\alpha_1^2+\beta^2)}\frac{\sin\gamma\alpha_1}{\alpha_1} -$$

$$\left. - \frac{2(-\alpha_1^2+\beta_1^2+2\beta_1\beta)\cos\gamma\alpha_1}{(\alpha_1^2+\beta_1^2)^3(\alpha_1^2+\beta^2)} - \frac{(-\alpha_1^2+\beta_1^2+2\beta_1\beta)\cos\gamma\alpha_1}{(\alpha_1^2+\beta_1^2)^2(\alpha_1^2+\beta^2)^2}\right)\cos\lambda u\, du +$$

$$+\int\limits_0^\infty e^{-\sigma(u^2+\alpha^2)}\left(\frac{-\sigma y_3(2\beta_1\alpha_1^2+\alpha_1^2\beta-\beta_1^2\beta)}{(\alpha_1^2+\beta_1^2)^2(\alpha_1^2+\beta^2)}\frac{\sin\gamma\alpha_1}{\alpha_1} + \frac{(2\beta_1+\beta)}{(\alpha_1^2+\beta_1^2)^2(\alpha_1^2+\beta^2)}\frac{\sin\tau\alpha}{\alpha_1} + \right.$$

$$+\frac{\sigma(2\beta_1\alpha_1^2+\alpha_1^2\beta-\beta_1^2\beta)}{(\alpha_1^2+\beta_1^2)^2(\alpha_1^2+\beta^2)}\frac{\cos\tau\alpha}{\alpha_1} - \frac{2(2\beta_1\alpha_1^2+\alpha_1^2\beta-\beta_1^2\beta)}{(\alpha_1^2+\beta_1^2)^3(\alpha_1^2+\beta^2)}\frac{\sin\gamma\alpha}{\alpha_1} -$$

$$\left.\left. - \frac{(2\beta_1\alpha_1^2+\alpha_1^2\beta-\beta_1^2\beta)}{(\alpha_1^2+\beta_1^2)^2(\alpha_1^2+\beta^2)^2}\frac{\sin\gamma\alpha}{\alpha_1} - \frac{(2\beta_1\alpha_1^2+\alpha_1^2\beta-\beta_1^2\beta)}{2\alpha_1^2(\alpha_1^2+\beta_1^2)^2(\alpha_1^2+\beta^2)}\frac{\sin\gamma\alpha}{\alpha_1}\right)\cos\lambda u\, du\right],$$

$$s = \alpha^2,\ \gamma = 2\sigma y_3.$$

Now, we estimate the integrals of $\int\limits_T \left|\frac{\partial \Phi_\sigma(y,x)}{\partial y_j}\right| ds_y,\ j = 1, 2.$

First we estimate $\int\limits_{y_3=0} \left|\frac{\Phi_\sigma(y,x)}{\partial y_1}\right| ds_y$. Given equality (32) and inequality (28), for $\alpha_1 \geq 1$ and $0 < \alpha_1 \leq 1$, we have

$$\int\limits_{y_3=0} \left|\frac{\Phi_\sigma(y,x)}{\partial y_1}\right| ds_y \leq C_\rho(x)\sigma\, e^{-\sigma x_3^2},\ \sigma > 1,\ x \in G, \tag{33}$$

$$\int\limits_{y_3=0} \left|\frac{\Phi_\sigma(y,x)}{\partial y_1}\right| ds_y \leq C_\rho(x)\sigma e^{-\sigma x_3^2},\ \sigma>1,\ x\in G. \tag{34}$$

Combining inequalities (33) and (34), we obtain

$$\int\limits_{y_3=0} \left|\frac{\Phi_\sigma(y,x)}{\partial y_1}\right| ds_y \leq C_\rho(x)\sigma e^{-\sigma x_3^2},\ \sigma>1,\ x\in G. \tag{35}$$

Now, we will similarly evaluate $\int\limits_{y_3=0} \left|\frac{\Phi_\sigma(y,x)}{\partial y_2}\right| ds_y$. Given equality (32) and inequality (28), for $\alpha_1 \geq 1$ and $0<\alpha_1\leq 1$, we have

$$\int\limits_{y_3=0} \left|\frac{\Phi_\sigma(y,x)}{\partial y_2}\right| ds_y \leq C_\rho(x)\sigma e^{-\sigma x_3^2},\ \sigma>1,\ x\in G, \tag{36}$$

$$\int\limits_{y_3=0} \left|\frac{\Phi_\sigma(y,x)}{\partial y_2}\right| ds_y \leq C_\rho(x)\sigma e^{-\sigma x_3^2},\ \sigma>1,\ x\in G. \tag{37}$$

Combining inequalities (36) and (37), we obtain

$$\int\limits_{y_3=0} \left|\frac{\Phi_\sigma(y,x)}{\partial y_2}\right| ds_y \leq C_\rho(x)\sigma e^{-\sigma x_3^2},\ \sigma>1,\ x\in G. \tag{38}$$

To estimate the integral $\int\limits_{y_3=0} \left|\frac{\Phi_\sigma(y,x)}{\partial y_3}\right| ds_y$, we use the equality

$$\begin{gathered}
\frac{\Phi_\sigma(y,x)}{\partial y_3} = \frac{2h^2 e^{\sigma(y_3^2-x_3^2)}}{\pi^2}\left[\int\limits_0^\infty e^{-\sigma(u^2+\alpha^2)}\left(\frac{2\sigma y_3(-\alpha_1^2+\beta_1^2+2\beta_1\beta)}{(\alpha_1^2+\beta_1^2)^2(\alpha_1^2+\beta^2)}\cos\gamma\alpha_1+\right.\right.\\
+\frac{(4\beta_1+2\beta)\cos\gamma\alpha_1}{(\alpha_1^2+\beta_1^2)^2(\alpha_1^2+\beta^2)}-\frac{2\sigma\alpha_1(-\alpha_1^2+\beta_1^2+2\beta_1\beta)\cos\gamma\alpha_1}{(\alpha_1^2+\beta_1^2)^2(\alpha_1^2+\beta^2)}-\\
\left.-\frac{4\beta_1(-\alpha_1^2+\beta_1^2+2\beta_1\beta)\cos\gamma\alpha_1}{(\alpha_1^2+\beta_1^2)^3(\alpha_1^2+\beta^2)}-\frac{2\beta(-\alpha_1^2+\beta_1^2+2\beta_1\beta)\cos\gamma\alpha_1}{(\alpha_1^2+\beta_1^2)^2(\alpha_1^2+\beta^2)^2}\right)\cos\lambda u\,du+\\
+\int\limits_0^\infty e^{-\sigma(u^2+\alpha^2)}\left(\frac{2\sigma y_3(2\alpha_1^2\beta_1+\alpha_1^2\beta-\beta_1^2\beta)}{(\alpha_1^2+\beta_1^2)^2(\alpha_1^2+\beta^2)}\frac{\sin\gamma\alpha_1}{\alpha_1}+\frac{(3\alpha_1^2-2\beta_1\beta-\beta_1^2)}{(\alpha_1^2+\beta_1^2)^2(\alpha_1^2+\beta^2)}\frac{\sin\gamma\alpha_1}{\alpha_1}+\right.\\
+\frac{2\sigma(2\alpha_1^2\beta_1+\alpha_1^2\beta-\beta_1^2\beta)\cos\gamma\alpha}{(\alpha_1^2+\beta_1^2)^3(\alpha_1^2+\beta^2)}-\frac{4\beta_1(2\alpha_1^2\beta_1+\alpha_1^2\beta-\beta_1^2\beta)}{(\alpha_1^2+\beta_1^2)^3(\alpha_1^2+\beta^2)}\frac{\sin\gamma\alpha_1}{\alpha_1}-\\
\left.\left.-\frac{2\beta(2\alpha_1^2\beta_1+\alpha_1^2\beta-\beta_1^2\beta)}{(\alpha_1^2+\beta_1^2)^2(\alpha_1^2+\beta^2)^2}\frac{\sin\gamma\alpha_1}{\alpha_1}\right)\cos\lambda u\,du\right],\ \gamma=2\sigma y_3.
\end{gathered} \tag{39}$$

Given equality (39) and inequality (28), for $\alpha_1\geq 1$ and $0<\alpha_1\leq 1$, we have

$$\int\limits_{y_3=0} \left|\frac{\Phi_\sigma(y,x)}{\partial y_3}\right| ds_y \leq C_\rho(x)\sigma e^{-\sigma x_3^2},\ \sigma>1,\ x\in G, \tag{40}$$

$$\int\limits_{y_3=0} \left|\frac{\Phi_\sigma(y,x)}{\partial y_3}\right| ds_y \leq C_\rho(x)\sigma e^{-\sigma x_3^2},\ \sigma > 1,\ x \in G. \tag{41}$$

Similarly, combining inequalities (40) - (41), we obtain

$$\int\limits_{y_3=0} \left|\frac{\Phi_\sigma(y,x)}{\partial y_3}\right| ds_y \leq C_\rho(x)\sigma e^{-\sigma x_3^2},\ \sigma > 1,\ x \in G. \tag{42}$$

From inequalities (31), (35), (38) and (42), we obtain

$$|U(x) - U_\sigma(x)| \leq C_\rho(x) M\sigma e^{-\sigma x_3^2},\ \sigma > 1,\ x \in G.$$

Theorem 3 is proved. □

Corollary 1. The limiting equality

$$\lim_{\sigma\to\infty} U_\sigma(x) = U(x),$$

holds uniformly on each compact set from the domain G.

Theorem 4. *Let $U(y) \in A_\rho(G)$ satisfy condition (24) on a part of the plane $y_3 = 0$, and inequality on a smooth surface S*

$$|U(y)| \leq \delta,\ 0 < \delta < 1. \tag{43}$$

Then the following estimate is correct

$$|U(x)| \leq C_\rho(x) M\sigma\delta^{\frac{x_3^2}{h^2}},\ \sigma > 1,\ x \in G. \tag{44}$$

Proof. Using the integral formula (22), we have

$$U(x) = \int\limits_{\partial G} N_\sigma(y,x)U(y)ds_y =$$

$$= \int\limits_S N_\sigma(y,x)U(y)ds_y + \int\limits_T N_\sigma(y,x)U(y)ds_y,\ x \in G.$$

Given the boundary condition (24) and inequality (43), we obtain the estimate

$$\begin{aligned} |U(x)| &\leq \int\limits_S |U(y)|\,|N_\sigma(y,x)|\,ds_y + \int\limits_T |U(y)|\,|N_\sigma(y,x)|\,ds_y \leq \\ &\leq \delta\int\limits_S |N_\sigma(y,x)|\,ds_y + M\int\limits_T |N_\sigma(y,x)|\,ds_y,\ x \in G. \end{aligned} \tag{45}$$

We first estimate the first integral of inequality (45). To do this, we estimate the integrals $\int\limits_S |\Phi_\sigma(y,x)|\,ds_y$, $\int_S \left|\frac{\partial\Phi_\sigma(y,x)}{\partial y_j}\right| ds_y$, $j = 1, 2$ and $\int\limits_S \left|\frac{\partial\Phi_\sigma(y,x)}{\partial y_3}\right| ds_y$ on a smooth surface S.

We first estimate $\int\limits_S |\Phi_\sigma(y,x)|\, ds_y$. Given equality (27) and inequality (28), for $\alpha_1 \geq 1$ and $0 < \alpha_1 \leq 1$, we have

$$\int\limits_S |\Phi_\sigma(y,x)|\, ds_y \leq C_\rho(x)\sigma\, e^{\sigma(h^2 - x_3^2)},\ \sigma > 1,\ x \in G, \tag{46}$$

$$\int\limits_S |\Phi_\sigma(y,x)|\, ds_y \leq C_\rho(x)\sigma\, e^{\sigma(h^2 - x_3^2)},\ \sigma > 1,\ x \in G. \tag{47}$$

Combining inequalities (46) and (47), we obtain

$$\int\limits_S |\Phi_\sigma(y,x)|\, ds_y \leq C_\rho(x)\sigma\, e^{\sigma(h^2 - x_3^2)},\ \sigma > 1,\ x \in G. \tag{48}$$

Now, we estimate the integrals of $\int\limits_S \left|\dfrac{\partial\Phi_\sigma(y,x)}{\partial y_j}\right| ds_y,\ j = 1, 2.$

First we estimate $\int\limits_S \left|\dfrac{\Phi_\sigma(y,x)}{\partial y_1}\right| ds_y$. Given equality (32) and inequality (28), for $\alpha_1 \geq 1$ and $0 < \alpha_1 \leq 1$, we have

$$\int\limits_S \left|\frac{\Phi_\sigma(y,x)}{\partial y_1}\right| ds_y \leq C_\rho(x)\sigma\, e^{\sigma(h^2 - x_3^2)},\ \sigma > 1,\ x \in G, \tag{49}$$

$$\int\limits_S \left|\frac{\Phi_\sigma(y,x)}{\partial y_1}\right| ds_y \leq C_\rho(x)\sigma\, e^{\sigma(h^2 - x_3^2)},\ \sigma > 1,\ x \in G. \tag{50}$$

Combining inequalities (49) and (50), we obtain

$$\int\limits_S \left|\frac{\Phi_\sigma(y,x)}{\partial y_1}\right| ds_y \leq C_\rho(x)\sigma\, e^{\sigma(h^2 - x_3^2)},\ \sigma > 1,\ x \in G. \tag{51}$$

Now, similarly we estimate $\int\limits_S \left|\dfrac{\Phi_\sigma(y,x)}{\partial y_2}\right| ds_y$. Given equality (32) and inequality (28), for $\alpha_1 \geq 1$ and $0 < \alpha_1 \leq 1$, we have

$$\int\limits_S \left|\frac{\Phi_\sigma(y,x)}{\partial y_2}\right| ds_y \leq C_\rho(x)\sigma\, e^{\sigma(h^2 - x_3^2)},\ \sigma > 1,\ x \in G, \tag{52}$$

$$\int\limits_S \left|\frac{\Phi_\sigma(y,x)}{\partial y_2}\right| ds_y \leq C_\rho(x)\sigma\, e^{\sigma(h^2 - x_3^2)},\ \sigma > 1,\ x \in G. \tag{53}$$

Combining inequalities (52) and (53), we obtain

$$\int\limits_S \left|\frac{\Phi_\sigma(y,x)}{\partial y_2}\right| ds_y \leq C_\rho(x)\sigma\, e^{\sigma(h^2 - x_3^2)},\ \sigma > 1,\ x \in G. \tag{54}$$

Now, we evaluate the integral $\int\limits_S \left|\dfrac{\partial\Phi_\sigma(y,x)}{\partial y_3}\right| ds_y$. Given equality (39) and inequality (28), for $\alpha_1 \geq 1$ and $0 < \alpha_1 \leq 1$, we have

$$\int\limits_S \left|\frac{\Phi_\sigma(y,x)}{\partial y_3}\right| ds_y \leq C_\rho(x)\sigma\, e^{\sigma(h^2 - x_3^2)},\ \sigma > 1,\ x \in G, \tag{55}$$

$$\int\limits_S \left|\frac{\Phi_\sigma(y,x)}{\partial y_3}\right| ds_y \leq C_\rho(x)\sigma\, e^{\sigma(h^2-x_3^2)},\ \sigma > 1,\ x \in G. \tag{56}$$

Similarly, combining inequalities (55) and (56), we obtain

$$\int\limits_S \left|\frac{\Phi_\sigma(y,x)}{\partial y_3}\right| ds_y \leq C_\rho(x)\sigma\, e^{\sigma(h^2-x_3^2)},\ \sigma > 1,\ x \in G. \tag{57}$$

From inequalities (48), (51), (54) and (57), we obtain

$$\delta\int\limits_S |N_\sigma(y,x)|\, ds_y \leq C_\rho(x)\sigma\delta\, e^{\sigma(h^2-x_3^2)},\ \sigma > 1,\ x \in G. \tag{58}$$

The following is known

$$M\int\limits_T |N_\sigma(y,x)|\, ds_y \leq C_\rho(x)M\sigma\, e^{-\sigma x_3^2},\ \sigma > 1,\ x \in G. \tag{59}$$

Now, taking into account (58) - (59), we have

$$|U(x)| \leq \frac{C_\rho(x)M\sigma}{2}(\delta\, e^{\sigma\, h^2}+1)e^{-\sigma\, x_3^2},\ \sigma > 1,\ x \in G.$$

Choosing σ from the equality

$$\sigma = \frac{1}{h^2}\ln\frac{1}{\delta}, \tag{60}$$

we obtain inequality (44). Theorem 4 is proved. □

Let $U(y) \in A_\rho(G)$ and instead of $U(y)$ on S be given its approximation $f_\delta(y)$, respectively, with an error $0 < \delta < 1$, $\max\limits_S |U(y) - f_\delta(y)| \leq \delta$.

We put

$$U_{\sigma(\delta)}(x) = \int\limits_S N_\sigma(y,x) f_\delta(y) ds_y,\ x \in G. \tag{61}$$

The following is true.

Theorem 5. *Let $U(y) \in A_\rho(G)$ satisfy condition (24) on a part of the plane $y_3 = 0$. Then the following estimate holds*

$$\left|U(x) - U_{\sigma(\delta)}(x)\right| \leq C_\rho(x)M\sigma\delta^{\frac{x_3^2}{h^2}},\ \sigma > 1,\ x \in G. \tag{62}$$

Proof. From the integral formulas (22) and (61), we have

$$U(x) - U_{\sigma(\delta)}(x) = \int\limits_S N_\sigma(y,x)\left\{U(y) - f_\delta(y)\right\} ds_y +$$

$$+\int\limits_T N_\sigma(y,x)U(y)ds_y,\ x \in G.$$

Now, repeating the proofs of Theorems 1 and 2, we obtain

$$\left|U(x) - U_{\sigma(\delta)}(x)\right| \leq \frac{C_\rho(x)M\sigma}{2}(\delta\, e^{\sigma\, h^2}+1)e^{-\sigma\, x_3^2},\ \ \sigma > 1,\ x \in G.$$

Hence, choosing σ from equality (60), we obtain (62). Theorem 5 is proved. □

Corollary 2. Limit equality

$$\lim_{\delta\to 0} U_{\sigma(\delta)}(x) = U(x),$$

holds uniformly on each compact set from the domain G.

4 Conclusion

In conclusion, we can say that on the basis of previous works (Juraev, 2012; Juraev, 2014; Juraev, 2017a; Juraev, 2017b; Juraev, 2017c; Juraev, 2018a; Juraev, 2018b; Juraev, 2018c; Juraev, 2019a; Juraev, 2019b), we constructed the Carleman matrix, and using this function, we found a regularized solution of the Cauchy problem for the matrix factorizations of the Helmholtz equation for a three-dimensional unbounded domain in an explicit form. Thus, the functional $U_{\sigma(\delta)}(x)$ determines the regularization of the solution of problem (1), (23).

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